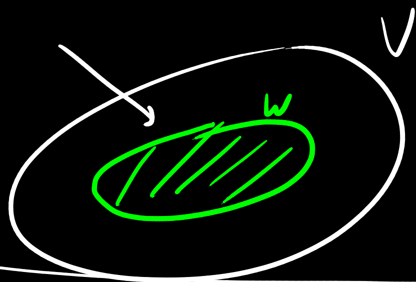
Last Time: Vector Subspaces.



Prop (Subspace Test):

Let V be a vector space and $W \subseteq V$. ($\subseteq$ subset.)

The [following are equivalent:]

1. $W \leq V$ i.e. W is a subspace of V
2. $0_V \in W$ and W is closed under the operations of V.
3. $W \neq \emptyset$ and for all $u, v \in W$ and all $r \in \mathbb{R}$ we have $u + r \cdot v \in W$.

Ex: Show $W = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\} \leq M_{2\times 2}(\mathbb{R})$.

Sol: We'll apply the subspace test!

To see $W \neq \emptyset$, we note $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$ (i.e. $a=b=c=0$ in the def^n of W)

Let $\begin{pmatrix} a_1 & 0 \\ b_1 & c_1 \end{pmatrix}$ and $\begin{pmatrix} a_2 & 0 \\ b_2 & c_2 \end{pmatrix}$ be elements of W and $r \in \mathbb{R}$.

Now
$$u + r v = \begin{pmatrix} a_1 & 0 \\ b_1 & c_1 \end{pmatrix} + r \cdot \begin{pmatrix} a_2 & 0 \\ b_2 & c_2 \end{pmatrix}$$
$$= \begin{pmatrix} a_1 & 0 \\ b_1 & c_1 \end{pmatrix} + \begin{pmatrix} ra_2 & r\cdot 0 \\ rb_2 & rc_2 \end{pmatrix}$$
$$= \begin{pmatrix} a_1 & 0 \\ b_1 & c_1 \end{pmatrix} + \begin{pmatrix} ra_2 & 0 \\ rb_2 & rc_2 \end{pmatrix}$$
$$= \begin{pmatrix} a_1 + ra_2 & 0+0 \\ b_1 + rb_2 & c_1 + rc_2 \end{pmatrix} = \begin{pmatrix} a_1 + ra_2 & 0 \\ b_1 + rb_2 & c_1 + rc_2 \end{pmatrix} \in W.$$

Hence $W \leq M_{2\times 2}(\mathbb{R})$ by the subspace test. $\blacksquare$

Span

Def^n: The span of subset $S \subseteq V$ of vector space V is the set of linear combinations of of elements from S. I.e.

$$\text{span}(S) = \left\{ a_1 s_1 + a_2 s_2 + \cdots + a_n s_n \;:\; \begin{array}{l} a_1, a_2, \dots, a_n \in \mathbb{R} \\ s_1, s_2, \dots, s_n \in S \end{array} \right\}.$$

Ex: Let $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$. Then

$$\begin{aligned} \text{Span}(S) &= \left\{ a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} : a_1, a_2 \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} a_1 - a_2 \\ a_1 + 2a_2 \end{bmatrix} : a_1, a_2 \in \mathbb{R} \right\}. \end{aligned}$$

$\boxtimes$

Fundamental Question: How do we decide if $v \in \text{Span}(S)$?

Ex: $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$. A vector $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ is in $\text{Span}(S)$ if and only if:

$$\rightarrow \quad a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\rightarrow \text{ i.e. } \begin{bmatrix} 1 \cdot a & -1 \cdot b \\ 1 \cdot a & +2 \cdot b \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\rightarrow \text{ i.e. } \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\rightarrow \text{ i.e. } \left[\begin{array}{cc|c} 1 & -1 & x \\ 1 & 2 & y \end{array}\right] \text{ has a solution.}$$

Let's symbolically solve $\left[\begin{array}{cc|c} 1 & -1 & x \\ 1 & 2 & y \end{array}\right]$:

$$\left[\begin{array}{cc|c} 1 & -1 & x \\ 1 & 2 & y \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & -1 & x \\ 0 & 3 & y-x \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & -1 & x \\ 0 & 1 & \frac{1}{3}(y-x) \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{2}{3}x + \frac{1}{3}y \\ 0 & 1 & \frac{1}{3}y - \frac{1}{3}x \end{array}\right]$$

$\therefore$ This system $\begin{cases} a - b = x \\ a + 2b = y \end{cases}$ has solution

$$a = \frac{2}{3}x + \frac{1}{3}y \quad \text{and} \quad b = \frac{1}{3}y - \frac{1}{3}x$$

Hence every $\begin{bmatrix} x \\ y \end{bmatrix}$ is in $\operatorname{Span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$

Hence $\operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \mathbb{R}^2$. $\boxtimes$

Ex: Compute $\operatorname{span}\{x^2+x+1, x^3-x\}$ in $P_3(\mathbb{R})$.

Sol: $\operatorname{span}\{x^2+x+1, x^3-x\} = \{a(x^2+x+1) + b(x^3-x) : a, b \in \mathbb{R}\}$

$$W = \{bx^3 + ax^2 + (a-b)x + a : a, b \in \mathbb{R}\}$$

Let's compute another parameterization of W.

$$s_3x^3 + s_2x^2 + s_1x + s_0 \in W$$

iff $a(x^2+x+1) + b(x^3-x) = s_3x^3 + s_2x^2 + s_1x + s_0$ for some $a, b \in \mathbb{R}$

iff $bx^3 + ax^2 + (a-b)x + a = s_3x^3 + s_2x^2 + s_1x + s_0$

$$\text{iff} \quad \begin{cases} b = s_3 \\ a = s_2 \\ a-b = s_1 \\ a = s_0 \end{cases} \rightsquigarrow \begin{array}{cc} a & b \\ \multicolumn{2}{c}{\left[\begin{array}{cc|c} 0 & 1 & s_3 \\ 1 & 0 & s_2 \\ 1 & -1 & s_1 \\ 1 & 0 & s_0 \end{array}\right]} \end{array} \quad \text{has a solution}$$

So we solve this (over determined) system:

Exercise: polys $\rightsquigarrow$ columns? How?

$$\left[\begin{array}{cc|c} 0 & 1 & s_3 \\ 1 & 0 & s_2 \\ 1 & -1 & s_1 \\ 1 & 0 & s_0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & s_2 \\ 0 & 1 & s_3 \\ 0 & -1 & s_1 - s_2 \\ 0 & 0 & s_0 - s_2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & s_2 \\ 0 & 1 & s_3 \\ 0 & 0 & s_1 - s_2 + s_3 \\ 0 & 0 & s_0 - s_2 \end{array}\right]$$

$$\text{Hence } W = \left\{ s_3 x^3 + s_2 x^2 + s_1 x + s_0 \; : \; \begin{array}{l} 0 = s_1 - s_2 + s_3 \\ 0 = s_0 - s_2 \end{array} \right\}$$

$$= \left\{ s_3 x^3 + s_2 x^2 + s_1 x + s_0 \; : \; s_2 = s_1 + s_3 = s_0 \right\}. \quad \boxtimes$$

Lem: Let $S \subseteq V$ be a subset of vector space V.

Then $\operatorname{span}(S) \leq V$.

(NB: $0_V \in \operatorname{Span}(S)$ for all S!).

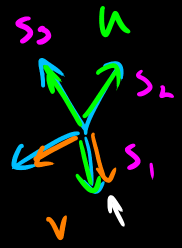
Convention: $\operatorname{Span}(\emptyset) = \operatorname{Span}(\{\}) = \{0_V\}$.

pf: Let $S \subseteq V$ be an arbitrary subset of V. We apply the subspace test. Notice $0_V \in \operatorname{Span}(S)$ automatically because 0_V is the empty sum over V.

Let $u, v \in \operatorname{span}(S)$ and $r \in \mathbb{R}$ be arbitrary.

Because $u, v \in \operatorname{span}(S)$, we may write

$$u = a_1 s_1 + a_2 s_2 + \cdots + a_n s_n$$
$$v = b_1 s_1 + b_2 s_2 + \cdots + b_n s_n + b_{n+1} s_{n+1} + \cdots + b_m s_m.$$



$u = a_1 s_1 + a_2 s_2 + a_3 s_3$
$v = b_1 s_1 + 0 s_2 + 0 s_3 + b_4 s_4$

Now adding $u + r\cdot v$ yields:

$$u + r\cdot v = (a_1 + rb_1) s_1 + (a_2 + rb_2) s_2 + \cdots + (a_n + rb_n) s_n + b_{n+1} s_{n+1} + \cdots + b_m s_m.$$

On the other hand, $a_i + rb_i \in \mathbb{R}$ so $u + r\cdot v$ is a linear combination of elements of S.

Hence $u + r\cdot v \in \operatorname{Span}(S)$ as desired. $\boxed{\checkmark}$

<u>Point</u>: Span takes a set of vectors and returns a subspace <u>determined by</u> them...

In particular, it turns out $\operatorname{span}(S)$ is the "smallest subspace of V containing S". $\boxed{\checkmark}$

<u>Ex</u>: Compute $\operatorname{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}\right\} =: W$.

<u>Sol</u>: $$W = \left\{ a\begin{bmatrix}1\\1\\1\end{bmatrix} + b\begin{bmatrix}2\\1\\0\end{bmatrix} + c\begin{bmatrix}3\\2\\1\end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

We have $\begin{bmatrix}x\\y\\z\end{bmatrix} \in W$ precisely when

$$a\begin{bmatrix}1\\1\\1\end{bmatrix} + b\begin{bmatrix}2\\1\\0\end{bmatrix} + c\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix}$$

i.e. $$\begin{bmatrix}a + 2b + 3c\\ a + b + 2c\\ a + c\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix}$$

i.e. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

i.e. $\left[\begin{array}{ccc|c} 1 & 2 & 3 & x \\ 1 & 1 & 2 & y \\ 1 & 0 & 1 & z \end{array}\right]$ has a solution. Solving:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & x \\ 1 & 1 & 2 & y \\ 1 & 0 & 1 & z \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & x \\ 0 & -1 & -1 & y-x \\ 0 & -2 & -2 & z-x \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & x \\ 0 & 1 & 1 & x-y \\ 0 & 0 & 0 & x-2y+z \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & -x-y \\ 0 & 1 & 1 & x-y \\ 0 & 0 & 0 & x-2y+z \end{array}\right]$$

$a + c = -x-y$, $b + c = x - y$,

$$\therefore\ W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x - 2y + z = 0 \right\}$$

NB: The set $S = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{u}, \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{v}, \underbrace{\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}}_{w} \right\}$ satisfies

$\boxed{u + v = w}$, i.e. w is a linear combination of u and v.

So: Actually $\operatorname{span}(S) = \operatorname{span}(S \setminus \{w\})$

i.e. $\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

$$= \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$